

UNITAL A_∞ -CATEGORIES

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ABSTRACT. We prove that three definitions of unitality for A_∞ -categories suggested by the first author, by Kontsevich and Soibelman, and by Fukaya are equivalent.

1. INTRODUCTION

Over the past decade, A_∞ -categories have experienced a resurgence of interest due to applications in symplectic geometry, deformation theory, non-commutative geometry, homological algebra, and physics.

The notion of A_∞ -category is a generalization of Stasheff's notion of A_∞ -algebra [12]. On the other hand, A_∞ -categories generalize differential graded categories. In contrast to differential graded categories, composition in A_∞ -categories is associative only up to homotopy that satisfies certain equation up to another homotopy, and so on. The notion of A_∞ -category appeared in the work of Fukaya on Floer homology [1] and was related to mirror symmetry by Kontsevich [5]. Basic concepts of the theory of A_∞ -categories have been developed by Fukaya [2], Keller [4], Lefèvre-Hasegawa [7], Lyubashenko [8], Soibelman [11].

The definition of A_∞ -category does not assume the existence of identity morphisms. The use of A_∞ -categories without identities requires caution: for example, there is no a sensible notion of isomorphic objects, the notion of equivalence does not make sense, etc. In order to develop a comprehensive theory of A_∞ -categories, a notion of unital A_∞ -category, i.e., A_∞ -category with identity morphisms (also called units), is necessary. The obvious notion of strictly unital A_∞ -category, despite its technical advantages, is not quite satisfactory: it is not homotopy invariant, meaning that it does not translate along homotopy equivalences. Different definitions of (weakly) unital A_∞ -category have been suggested by the first author [8, Definition 7.3], by Kontsevich and Soibelman [6, Definition 4.2.3], and by Fukaya [2, Definition 5.11]. We prove that these definitions are equivalent. The main ingredient of the proofs is the Yoneda Lemma for unital (in the sense of Lyubashenko) A_∞ -categories proven in [9, Appendix A], see also [10, Appendix A].

2. PRELIMINARIES

We follow the notation and conventions of [8], sometimes without explicit mentioning. Some of the conventions are recalled here.

Throughout, \mathbb{k} is a commutative ground ring. A graded \mathbb{k} -module always means a \mathbb{Z} -graded \mathbb{k} -module.

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A *graded quiver* \mathcal{A} consists of a set $\text{Ob } \mathcal{A}$ of objects and a graded \mathbb{k} -module $\mathcal{A}(X, Y)$, for each $X, Y \in \text{Ob } \mathcal{A}$. A *morphism of graded quivers* $f : \mathcal{A} \rightarrow \mathcal{B}$ of degree n consists of a function $\text{Ob } f : \text{Ob } \mathcal{A} \rightarrow \text{Ob } \mathcal{B}$, $X \mapsto Xf$, and a \mathbb{k} -linear map $f = f_{X,Y} : \mathcal{A}(X, Y) \rightarrow \mathcal{B}(Xf, Yf)$ of degree n , for each $X, Y \in \text{Ob } \mathcal{A}$.

For a set S , there is a category \mathcal{Q}/S defined as follows. Its objects are graded quivers whose set of objects is S . A morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ in \mathcal{Q}/S is a morphism of graded quivers of degree 0 such that $\text{Ob } f = \text{id}_S$. The category \mathcal{Q}/S is monoidal. The tensor product of graded quivers \mathcal{A} and \mathcal{B} is a graded quiver $\mathcal{A} \otimes \mathcal{B}$ such that

$$(\mathcal{A} \otimes \mathcal{B})(X, Z) = \bigoplus_{Y \in S} \mathcal{A}(X, Y) \otimes \mathcal{B}(Y, Z), \quad X, Z \in S.$$

The unit object is the *discrete quiver* $\mathbb{k}S$ with $\text{Ob } \mathbb{k}S = S$ and

$$(\mathbb{k}S)(X, Y) = \begin{cases} \mathbb{k} & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases} \quad X, Y \in S.$$

Note that a map of sets $f : S \rightarrow R$ gives rise to a morphism of graded quivers $\mathbb{k}f : \mathbb{k}S \rightarrow \mathbb{k}R$ with $\text{Ob } \mathbb{k}f = f$ and $(\mathbb{k}f)_{X,Y} = \text{id}_{\mathbb{k}}$ if $X = Y$ and $(\mathbb{k}f)_{X,Y} = 0$ if $X \neq Y$, $X, Y \in S$.

An *augmented graded cocategory* is a graded quiver \mathcal{C} equipped with the structure of an augmented counital coassociative coalgebra in the monoidal category $\mathcal{Q}/\text{Ob } \mathcal{C}$. Thus, \mathcal{C} comes with a comultiplication $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$, a counit $\varepsilon : \mathcal{C} \rightarrow \mathbb{k} \text{Ob } \mathcal{C}$, and an augmentation $\eta : \mathbb{k} \text{Ob } \mathcal{C} \rightarrow \mathcal{C}$, which are morphisms in $\mathcal{Q}/\text{Ob } \mathcal{C}$ satisfying the usual axioms. A *morphism of augmented graded cocategories* $f : \mathcal{C} \rightarrow \mathcal{D}$ is a morphism of graded quivers of degree 0 that preserves the comultiplication, counit, and augmentation.

The main example of an augmented graded cocategory is the following. Let \mathcal{A} be a graded quiver. Denote by $T\mathcal{A}$ the direct sum of graded quivers $T^n\mathcal{A}$, where $T^n\mathcal{A} = \mathcal{A}^{\otimes n}$ is the n -fold tensor product of \mathcal{A} in $\mathcal{Q}/\text{Ob } \mathcal{A}$; in particular, $T^0\mathcal{A} = \mathbb{k} \text{Ob } \mathcal{A}$, $T^1\mathcal{A} = \mathcal{A}$, $T^2\mathcal{A} = \mathcal{A} \otimes \mathcal{A}$, etc. The graded quiver $T\mathcal{A}$ is an augmented graded cocategory in which the comultiplication is the so called ‘cut’ comultiplication $\Delta_0 : T\mathcal{A} \rightarrow T\mathcal{A} \otimes T\mathcal{A}$ given by

$$f_1 \otimes \cdots \otimes f_n \mapsto \sum_{k=0}^n f_1 \otimes \cdots \otimes f_k \bigotimes f_{k+1} \otimes \cdots \otimes f_n,$$

the counit is given by the projection $\text{pr}_0 : T\mathcal{A} \rightarrow T^0\mathcal{A} = \mathbb{k} \text{Ob } \mathcal{A}$, and the augmentation is given by the inclusion $\text{in}_0 : \mathbb{k} \text{Ob } \mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$.

The graded quiver $T\mathcal{A}$ admits also the structure of a graded category, i.e., the structure of a unital associative algebra in the monoidal category $\mathcal{Q}/\text{Ob } \mathcal{A}$. The multiplication $\mu : T\mathcal{A} \otimes T\mathcal{A} \rightarrow T\mathcal{A}$ removes brackets in tensors of the form $(f_1 \otimes \cdots \otimes f_m) \bigotimes (g_1 \otimes \cdots \otimes g_n)$. The unit $\eta : \mathbb{k} \text{Ob } \mathcal{A} \rightarrow T\mathcal{A}$ is given by the inclusion $\text{in}_0 : \mathbb{k} \text{Ob } \mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$.

For a graded quiver \mathcal{A} , denote by $s\mathcal{A}$ its *suspension*, the graded quiver given by $\text{Ob } s\mathcal{A} = \text{Ob } \mathcal{A}$ and $(s\mathcal{A}(X, Y))^n = \mathcal{A}(X, Y)^{n+1}$, for each $n \in \mathbb{Z}$ and $X, Y \in \text{Ob } \mathcal{A}$. An A_∞ -category is a graded quiver \mathcal{A} equipped with a differential $b : Ts\mathcal{A} \rightarrow Ts\mathcal{A}$ of degree 1 such that $(Ts\mathcal{A}, \Delta_0, \text{pr}_0, \text{in}_0, b)$ is an *augmented differential graded cocategory*. In other terms, the equations

$$b^2 = 0, \quad b\Delta_0 = \Delta_0(b \otimes 1 + 1 \otimes b), \quad b\text{pr}_0 = 0, \quad \text{in}_0 b = 0$$

hold true. Denote by

$$b_{mn} \stackrel{\text{def}}{=} [T^m s\mathcal{A} \xrightarrow{\text{in}_m} Ts\mathcal{A} \xrightarrow{b} Ts\mathcal{A} \xrightarrow{\text{pr}_n} T^n s\mathcal{A}]$$

matrix coefficients of b , for $m, n \geq 0$. Matrix coefficients b_{m1} are called *components* of b and abbreviated by b_m . The above equations imply that $b_0 = 0$ and that b is unambiguously determined by its components via the formula

$$b_{mn} = \sum_{\substack{p+k+q=m \\ p+1+q=n}} 1^{\otimes p} \otimes b_k \otimes 1^{\otimes q} : T^m s\mathcal{A} \rightarrow T^n s\mathcal{A}, \quad m, n \geq 0.$$

The equation $b^2 = 0$ is equivalent to the system of equations

$$\sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q}) b_{p+1+q} = 0 : T^m s\mathcal{A} \rightarrow s\mathcal{A}, \quad m \geq 1.$$

For A_∞ -categories \mathcal{A} and \mathcal{B} , an A_∞ -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of augmented differential graded cocategories $f : Ts\mathcal{A} \rightarrow Ts\mathcal{B}$. In other terms, f is a morphism of augmented graded cocategories and preserves the differential, meaning that $fb = bf$. Denote by

$$f_{mn} \stackrel{\text{def}}{=} [T^m s\mathcal{A} \xrightarrow{\text{in}_m} Ts\mathcal{A} \xrightarrow{f} Ts\mathcal{B} \xrightarrow{\text{pr}_n} T^n s\mathcal{B}]$$

matrix coefficients of f , for $m, n \geq 0$. Matrix coefficients f_{m1} are called *components* of f and abbreviated by f_m . The condition that f is a morphism of augmented graded cocategories implies that $f_0 = 0$ and that f is unambiguously determined by its components via the formula

$$f_{mn} = \sum_{i_1+\dots+i_n=m} f_{i_1} \otimes \dots \otimes f_{i_n} : T^m s\mathcal{A} \rightarrow T^n s\mathcal{B}, \quad m, n \geq 0.$$

The equation $fb = bf$ is equivalent to the system of equations

$$\sum_{i_1+\dots+i_n=m} (f_{i_1} \otimes \dots \otimes f_{i_n}) b_n = \sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q}) f_{p+1+q} : T^m s\mathcal{A} \rightarrow s\mathcal{B},$$

for $m \geq 1$. An A_∞ -functor f is called *strict* if $f_n = 0$ for $n > 1$.

3. DEFINITIONS

3.1. Definition (cf. [2, 4]). An A_∞ -category \mathcal{A} is *strictly unital* if, for each $X \in \text{Ob } \mathcal{A}$, there is a \mathbb{k} -linear map ${}_X \mathbf{i}_0^A : \mathbb{k} \rightarrow (s\mathcal{A})^{-1}(X, X)$, called a *strict unit*, such that the following conditions are satisfied: ${}_X \mathbf{i}_0^A b_1 = 0$, the chain maps $(1 \otimes_Y \mathbf{i}_0^A) b_2, -({}_X \mathbf{i}_0^A \otimes 1) b_2 : s\mathcal{A}(X, Y) \rightarrow s\mathcal{A}(X, Y)$ are equal to the identity map, for each $X, Y \in \text{Ob } \mathcal{A}$, and $(\dots \otimes \mathbf{i}_0^A \otimes \dots) b_n = 0$ if $n \geq 3$.

For example, differential graded categories are strictly unital.

3.2. Definition (Lyubashenko [8, Definition 7.3]). An A_∞ -category \mathcal{A} is *unital* if, for each $X \in \text{Ob } \mathcal{A}$, there is a \mathbb{k} -linear map ${}_X \mathbf{i}_0^A : \mathbb{k} \rightarrow (s\mathcal{A})^{-1}(X, X)$, called a *unit*, such that the following conditions hold: ${}_X \mathbf{i}_0^A b_1 = 0$ and the chain maps $(1 \otimes_Y \mathbf{i}_0^A) b_2, -({}_X \mathbf{i}_0^A \otimes 1) b_2 : s\mathcal{A}(X, Y) \rightarrow s\mathcal{A}(X, Y)$ are homotopic to the identity map, for each $X, Y \in \text{Ob } \mathcal{A}$. An arbitrary homotopy between $(1 \otimes_Y \mathbf{i}_0^A) b_2$ and the identity map is called a *right unit homotopy*. Similarly, an arbitrary homotopy between $-({}_X \mathbf{i}_0^A \otimes 1) b_2$ and the identity map

is called a *left unit homotopy*. An A_∞ -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ between unital A_∞ -categories is *unital* if the cycles ${}_X \mathbf{i}_0^{\mathcal{A}} f_1$ and ${}_X f \mathbf{i}_0^{\mathcal{B}}$ are cohomologous, i.e., differ by a boundary, for each $X \in \text{Ob } \mathcal{A}$.

Clearly, a strictly unital A_∞ -category is unital.

With an arbitrary A_∞ -category \mathcal{A} a strictly unital A_∞ -category \mathcal{A}^{su} with the same set of objects is associated. For each $X, Y \in \text{Ob } \mathcal{A}$, the graded \mathbb{k} -module $s\mathcal{A}^{\text{su}}(X, Y)$ is given by

$$s\mathcal{A}^{\text{su}}(X, Y) = \begin{cases} s\mathcal{A}(X, Y) & \text{if } X \neq Y, \\ s\mathcal{A}(X, X) \oplus \mathbb{k}_X \mathbf{i}_0^{\mathcal{A}^{\text{su}}} & \text{if } X = Y, \end{cases}$$

where ${}_X \mathbf{i}_0^{\mathcal{A}^{\text{su}}}$ is a new generator of degree -1 . The element ${}_X \mathbf{i}_0^{\mathcal{A}^{\text{su}}}$ is a strict unit by definition, and the natural embedding $e : \mathcal{A} \hookrightarrow \mathcal{A}^{\text{su}}$ is a strict A_∞ -functor.

3.3. Definition (Kontsevich–Soibelman [6, Definition 4.2.3]). A *weak unit* of an A_∞ -category \mathcal{A} is an A_∞ -functor $U : \mathcal{A}^{\text{su}} \rightarrow \mathcal{A}$ such that

$$[\mathcal{A} \xrightarrow{e} \mathcal{A}^{\text{su}} \xrightarrow{U} \mathcal{A}] = \text{id}_{\mathcal{A}}.$$

3.4. Proposition. *Suppose that an A_∞ -category \mathcal{A} admits a weak unit. Then the A_∞ -category \mathcal{A} is unital.*

Proof. Let $U : \mathcal{A}^{\text{su}} \rightarrow \mathcal{A}$ be a weak unit of \mathcal{A} . For each $X \in \text{Ob } \mathcal{A}$, denote by ${}_X \mathbf{i}_0^{\mathcal{A}}$ the element ${}_X \mathbf{i}_0^{\mathcal{A}^{\text{su}}} U_1 \in s\mathcal{A}(X, X)$ of degree -1 . It follows from the equation $U_1 b_1 = b_1 U_1$ that ${}_X \mathbf{i}_0^{\mathcal{A}} b_1 = 0$. Let us prove that ${}_X \mathbf{i}_0^{\mathcal{A}}$ are unit elements of \mathcal{A} .

For each $X, Y \in \text{Ob } \mathcal{A}$, there is a \mathbb{k} -linear map

$$h = (1 \otimes {}_Y \mathbf{i}_0) U_2 : s\mathcal{A}(X, Y) \rightarrow s\mathcal{A}(X, Y)$$

of degree -1 . The equation

$$(1 \otimes b_1 + b_1 \otimes 1) U_2 + b_2 U_1 = U_2 b_1 + (U_1 \otimes U_1) b_2 \quad (3.1)$$

implies that

$$-b_1 h + 1 = h b_1 + (1 \otimes {}_Y \mathbf{i}_0^{\mathcal{A}}) b_2 : s\mathcal{A}(X, Y) \rightarrow s\mathcal{A}(X, Y),$$

thus h is a right unit homotopy for \mathcal{A} . For each $X, Y \in \text{Ob } \mathcal{A}$, there is a \mathbb{k} -linear map

$$h' = -({}_X \mathbf{i}_0 \otimes 1) U_2 : s\mathcal{A}(X, Y) \rightarrow s\mathcal{A}(X, Y)$$

of degree -1 . Equation (3.1) implies that

$$b_1 h' - 1 = -h' b_1 + ({}_X \mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2 : s\mathcal{A}(X, Y) \rightarrow s\mathcal{A}(X, Y),$$

thus h' is a left unit homotopy for \mathcal{A} . Therefore, \mathcal{A} is unital. \square

3.5. Definition (Fukaya [2, Definition 5.11]). An A_∞ -category \mathcal{C} is called *homotopy unital* if the graded quiver

$$\mathcal{C}^+ = \mathcal{C} \oplus \mathbb{k}\mathcal{C} \oplus s\mathbb{k}\mathcal{C}$$

(with $\text{Ob } \mathcal{C}^+ = \text{Ob } \mathcal{C}$) admits an A_∞ -structure b^+ of the following kind. Denote the generators of the second and the third direct summands of the graded quiver $s\mathcal{C}^+ = s\mathcal{C} \oplus s\mathbb{k}\mathcal{C} \oplus s^2\mathbb{k}\mathcal{C}$ by ${}_X \mathbf{i}_0^{\mathcal{C}^+} = 1s$ and $\mathbf{j}_X^{\mathcal{C}^+} = 1s^2$ of degree respectively -1 and -2 , for each $X \in \text{Ob } \mathcal{C}$. The conditions on b^+ are:

- (1) for each $X \in \text{Ob } \mathcal{C}$, the element ${}_X \mathbf{i}_0^{\mathcal{C}} \stackrel{\text{def}}{=} {}_X \mathbf{i}_0^{\mathcal{C}^+} - \mathbf{j}_X^{\mathcal{C}^+} b_1^+$ is contained in $s\mathcal{C}(X, X)$;

- (2) \mathcal{C}^+ is a strictly unital A_∞ -category with strict units ${}_X \mathbf{i}_0^{\mathcal{C}^{\text{su}}}$, $X \in \text{Ob } \mathcal{C}$;
- (3) the embedding $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is a strict A_∞ -functor;
- (4) $(s\mathcal{C} \oplus s^2 \mathbb{k}\mathcal{C})^{\otimes n} b_n^+ \subset s\mathcal{C}$, for each $n > 1$.

In particular, \mathcal{C}^+ contains the strictly unital A_∞ -category $\mathcal{C}^{\text{su}} = \mathcal{C} \oplus \mathbb{k}\mathcal{C}$. A version of this definition suitable for filtered A_∞ -algebras (and filtered A_∞ -categories) is given by Fukaya, Oh, Ohta and Ono in their book [3, Definition 8.2].

Let \mathcal{D} be a strictly unital A_∞ -category with strict units $\mathbf{i}_0^{\mathcal{D}}$. Then it has a canonical homotopy unital structure (\mathcal{D}^+, b^+) . Namely, $\mathbf{j}_X^{\mathcal{D}} b_1^+ = {}_X \mathbf{i}_0^{\mathcal{D}^{\text{su}}} - {}_X \mathbf{i}_0^{\mathcal{D}}$, and b_n^+ vanishes for each $n > 1$ on each summand of $(s\mathcal{D} \oplus s^2 \mathbb{k}\mathcal{D})^{\otimes n}$ except on $s\mathcal{D}^{\otimes n}$, where it coincides with $b_n^{\mathcal{D}}$. Verification of the equation $(b^+)^2 = 0$ is a straightforward computation.

3.6. Proposition. *An arbitrary homotopy unital A_∞ -category is unital.*

Proof. Let $\mathcal{C} \subset \mathcal{C}^+$ be a homotopy unital category. We claim that the distinguished cycles ${}_X \mathbf{i}_0^{\mathcal{C}} \in \mathcal{C}(X, X)[1]^{-1}$, $X \in \text{Ob } \mathcal{C}$, turn \mathcal{C} into a unital A_∞ -category. Indeed, the identity

$$(1 \otimes b_1^+ + b_1^+ \otimes 1)b_2^+ + b_2^+ b_1^+ = 0$$

applied to $s\mathcal{C} \otimes \mathbf{j}^{\mathcal{C}}$ or to $\mathbf{j}^{\mathcal{C}} \otimes s\mathcal{C}$ implies

$$\begin{aligned} (1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2^{\mathcal{C}} &= 1 + (1 \otimes \mathbf{j}^{\mathcal{C}})b_2^+ b_1^{\mathcal{C}} + b_1^{\mathcal{C}}(1 \otimes \mathbf{j}^{\mathcal{C}})b_2^+ : s\mathcal{C} \rightarrow s\mathcal{C}, \\ (\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2^{\mathcal{C}} &= -1 + (\mathbf{j}^{\mathcal{C}} \otimes 1)b_2^+ b_1^{\mathcal{C}} + b_1^{\mathcal{C}}(\mathbf{j}^{\mathcal{C}} \otimes 1)b_2^+ : s\mathcal{C} \rightarrow s\mathcal{C}. \end{aligned}$$

Thus, $(1 \otimes \mathbf{j}^{\mathcal{C}})b_2^+ : s\mathcal{C} \rightarrow s\mathcal{C}$ and $(\mathbf{j}^{\mathcal{C}} \otimes 1)b_2^+ : s\mathcal{C} \rightarrow s\mathcal{C}$ are unit homotopies. Therefore, the A_∞ -category \mathcal{C} is unital. \square

The converse of Proposition 3.6 holds true as well.

3.7. Theorem. *An arbitrary unital A_∞ -category \mathcal{C} with unit elements $\mathbf{i}_0^{\mathcal{C}}$ admits a homotopy unital structure (\mathcal{C}^+, b^+) with $\mathbf{j}^{\mathcal{C}} b_1^+ = \mathbf{i}_0^{\mathcal{C}^{\text{su}}} - \mathbf{i}_0^{\mathcal{C}}$.*

Proof. By [10, Corollary A.12], there exists a differential graded category \mathcal{D} and an A_∞ -equivalence $\phi : \mathcal{C} \rightarrow \mathcal{D}$. By [10, Remark A.13], we may choose \mathcal{D} and ϕ such that $\text{Ob } \mathcal{D} = \text{Ob } \mathcal{C}$ and $\text{Ob } \phi = \text{id}_{\text{Ob } \mathcal{C}}$. Being strictly unital \mathcal{D} admits a canonical homotopy unital structure (\mathcal{D}^+, b^+) . In the sequel, we may assume that \mathcal{D} is a strictly unital A_∞ -category equivalent to \mathcal{C} via ϕ with the mentioned properties. Let us construct simultaneously an A_∞ -structure b^+ on \mathcal{C}^+ and an A_∞ -functor $\phi^+ : \mathcal{C}^+ \rightarrow \mathcal{D}^+$ that will turn out to be an equivalence.

Let us extend the homotopy isomorphism $\phi_1 : s\mathcal{C} \rightarrow s\mathcal{D}$ to a chain quiver map $\phi_1^+ : s\mathcal{C}^+ \rightarrow s\mathcal{D}^+$. The A_∞ -equivalence $\phi : \mathcal{C} \rightarrow \mathcal{D}$ is a unital A_∞ -functor, i.e., for each $X \in \text{Ob } \mathcal{C}$, there exists $v_X \in \mathcal{D}(X, X)[1]^{-2}$ such that ${}_X \mathbf{i}_0^{\mathcal{D}} - {}_X \mathbf{i}_0^{\mathcal{C}} \phi_1 = v_X b_1$. In order that ϕ^+ be strictly unital, we define ${}_X \mathbf{i}_0^{\mathcal{C}^{\text{su}}} \phi_1^+ = {}_X \mathbf{i}_0^{\mathcal{D}^{\text{su}}}$. We should have

$$\mathbf{j}_X^{\mathcal{C}} \phi_1^+ b_1^+ = \mathbf{j}_X^{\mathcal{C}} b_1^+ \phi_1^+ = {}_X \mathbf{i}_0^{\mathcal{C}^{\text{su}}} \phi_1^+ - {}_X \mathbf{i}_0^{\mathcal{C}} \phi_1 = {}_X \mathbf{i}_0^{\mathcal{D}^{\text{su}}} - {}_X \mathbf{i}_0^{\mathcal{D}} + {}_X \mathbf{i}_0^{\mathcal{D}} - {}_X \mathbf{i}_0^{\mathcal{C}} \phi_1 = (\mathbf{j}_X^{\mathcal{D}} + v_X) b_1^+,$$

so we define $\mathbf{j}_X^{\mathcal{C}} \phi_1^+ = \mathbf{j}_X^{\mathcal{D}} + v_X$.

We claim that there is a homotopy unital structure (\mathcal{C}^+, b^+) of \mathcal{C} satisfying the four conditions of Definition 3.5 and an A_∞ -functor $\phi^+ : \mathcal{C}^+ \rightarrow \mathcal{D}^+$ satisfying four parallel conditions:

- (1) the first component of ϕ^+ is the quiver morphism ϕ_1^+ constructed above;

- (2) the A_∞ -functor ϕ^+ is strictly unital;
- (3) the restriction of ϕ^+ to \mathcal{C} gives ϕ ;
- (4) $(s\mathcal{C} \oplus s^2\mathbb{k}\mathcal{C})^{\otimes n} \phi_n^+ \subset s\mathcal{D}$, for each $n > 1$.

Notice that in the presence of conditions (2) and (3) the first condition reduces to $\mathbf{j}_X^{\mathcal{C}}(\phi^+)_1 = \mathbf{j}_X^{\mathcal{D}} + v_X$, for each $X \in \text{Ob } \mathcal{C}$.

Components of the (1,1)-coderivation $b^+ : Ts\mathcal{C}^+ \rightarrow Ts\mathcal{C}^+$ of degree 1 and of the augmented graded cocategory morphism $\phi^+ : Ts\mathcal{C}^+ \rightarrow Ts\mathcal{D}^+$ are constructed by induction. We already know components b_1^+ and ϕ_1^+ . Given an integer $n \geq 2$, assume that we have already found components b_m^+ , ϕ_m^+ of the sought b^+ and ϕ^+ for $m < n$ such that the equations

$$((b^+)^2)_m = 0 : T^m s\mathcal{C}^+(X, Y) \rightarrow s\mathcal{C}^+(X, Y), \quad (3.2)$$

$$(\phi^+ b^+)_m = (b^+ \phi^+)_m : T^m s\mathcal{C}^+(X, Y) \rightarrow s\mathcal{D}^+(Xf, Yf) \quad (3.3)$$

are satisfied for all $m < n$. Define b_n^+ , ϕ_n^+ on direct summands of $T^n s\mathcal{C}^+$ which contain a factor $\mathbf{i}_0^{\text{csu}}$ by the requirement of strict unitality of \mathcal{C}^+ and ϕ^+ . Then equations (3.2), (3.3) hold true for $m = n$ on such summands. Define b_n^+ , ϕ_n^+ on the direct summand $T^n s\mathcal{C} \subset T^n s\mathcal{C}^+$ as $b_n^{\mathcal{C}}$ and ϕ_n . Then equations (3.2), (3.3) hold true for $m = n$ on the summand $T^n s\mathcal{C}$. It remains to construct those components of b^+ and ϕ^+ which have $\mathbf{j}^{\mathcal{C}}$ as one of their arguments.

Extend $b_1 : s\mathcal{C} \rightarrow s\mathcal{C}$ to $b_1' : s\mathcal{C}^+ \rightarrow s\mathcal{C}^+$ by $\mathbf{i}_0^{\text{csu}} b_1' = 0$ and $\mathbf{j}^{\mathcal{C}} b_1' = 0$. Define $b_1^- = b_1^+ - b_1' : s\mathcal{C}^+ \rightarrow s\mathcal{C}^+$. Thus, $b_1^-|_{s\mathcal{C}^{\text{su}}} = 0$, $\mathbf{j}^{\mathcal{C}} b_1^- = \mathbf{i}_0^{\text{csu}} - \mathbf{i}_0^{\mathcal{C}}$ and $b_1^+ = b_1' + b_1^-$. Introduce for $0 \leq k \leq n$ the graded subquiver $\mathcal{N}_k \subset T^n(s\mathcal{C} \oplus s^2\mathbb{k}\mathcal{C})$ by

$$\mathcal{N}_k = \bigoplus_{p_0+p_1+\dots+p_k+k=n} T^{p_0} s\mathcal{C} \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_1} s\mathcal{C} \otimes \dots \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_k} s\mathcal{C}$$

stable under the differential $d^{\mathcal{N}_k} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b_1' \otimes 1^{\otimes q}$, and the graded subquiver $\mathcal{P}_l \subset T^n s\mathcal{C}^+$ by

$$\mathcal{P}_l = \bigoplus_{p_0+p_1+\dots+p_l+l=n} T^{p_0} s\mathcal{C}^{\text{su}} \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_1} s\mathcal{C}^{\text{su}} \otimes \dots \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_l} s\mathcal{C}^{\text{su}}.$$

There is also the subquiver

$$\mathcal{Q}_k = \bigoplus_{l=0}^k \mathcal{P}_l \subset T^n s\mathcal{C}^+$$

and its complement

$$\mathcal{Q}_k^\perp = \bigoplus_{l=k+1}^n \mathcal{P}_l \subset T^n s\mathcal{C}^+.$$

Notice that the subquiver \mathcal{Q}_k is stable under the differential $d^{\mathcal{Q}_k} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b_1^+ \otimes 1^{\otimes q}$, and \mathcal{Q}_k^\perp is stable under the differential $d^{\mathcal{Q}_k^\perp} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b_1' \otimes 1^{\otimes q}$. Furthermore, the image of $1^{\otimes a} \otimes b_1^- \otimes 1^{\otimes c} : \mathcal{N}_k \rightarrow T^n s\mathcal{C}^+$ is contained in \mathcal{Q}_{k-1} for all $a, c \geq 0$ such that $a + 1 + c = n$.

Firstly, the components b_n^+ , ϕ_n^+ are defined on the graded subquivers $\mathcal{N}_0 = T^n s\mathcal{C}$ and $\mathcal{Q}_0 = T^n s\mathcal{C}^{\text{su}}$. Assume for an integer $0 < k \leq n$ that restrictions of b_n^+ , ϕ_n^+ to \mathcal{N}_l are already

found for all $l < k$. In other terms, we are given $b_n^+ : \mathcal{Q}_{k-1} \rightarrow s\mathcal{C}^+$, $\phi_n^+ : \mathcal{Q}_{k-1} \rightarrow s\mathcal{D}$ such that equations (3.2), (3.3) hold on \mathcal{Q}_{k-1} . Let us construct the restrictions $b_n^+ : \mathcal{N}_k \rightarrow s\mathcal{C}$, $\phi_n^+ : \mathcal{N}_k \rightarrow s\mathcal{D}$, performing the induction step.

Introduce a $(1,1)$ -coderivation $\tilde{b} : Ts\mathcal{C}^+ \rightarrow Ts\mathcal{C}^+$ of degree 1 by its components $(0, b_1^+, \dots, b_{n-1}^+, \text{pr}_{\mathcal{Q}_{k-1}} \cdot b_n^+|_{\mathcal{Q}_{k-1}}, 0, \dots)$. Introduce also a morphism of augmented graded cocategories $\tilde{\phi} : Ts\mathcal{C}^+ \rightarrow Ts\mathcal{D}^+$ with $\text{Ob } \tilde{\phi} = \text{Ob } \phi$ by its components

$$(\phi_1^+, \dots, \phi_{n-1}^+, \text{pr}_{\mathcal{Q}_{k-1}} \cdot \phi_n^+|_{\mathcal{Q}_{k-1}}, 0, \dots).$$

Here $\text{pr}_{\mathcal{Q}_{k-1}} : T^n s\mathcal{C}^+ \rightarrow \mathcal{Q}_{k-1}$ is the natural projection, vanishing on \mathcal{Q}_{k-1}^\perp . Then $\lambda \stackrel{\text{def}}{=} \tilde{b}^2 : Ts\mathcal{C}^+ \rightarrow Ts\mathcal{C}^+$ is a $(1,1)$ -coderivation of degree 2 and $\nu \stackrel{\text{def}}{=} -\tilde{\phi}b^+ + \tilde{b}\tilde{\phi} : Ts\mathcal{C}^+ \rightarrow Ts\mathcal{D}^+$ is a $(\tilde{\phi}, \tilde{\phi})$ -coderivation of degree 1. Equations (3.2), (3.3) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$. Moreover, λ_n, ν_n vanish on \mathcal{Q}_{k-1} . On the complement the n -th components equal

$$\begin{aligned} \lambda_n &= \sum_{\substack{1 < r < n \\ a+r+c=n}} (1^{\otimes a} \otimes b_r^+ \otimes 1^{\otimes c}) b_{a+1+c}^+ + \sum_{a+1+c=n} (1^{\otimes a} \otimes b_1^- \otimes 1^{\otimes c}) \tilde{b}_n : \mathcal{Q}_{k-1}^\perp \rightarrow s\mathcal{C}^+, \\ \nu_n &= - \sum_{\substack{1 < r \leq n \\ i_1 + \dots + i_r = n}} (\phi_{i_1}^+ \otimes \dots \otimes \phi_{i_r}^+) b_r^+ + \sum_{\substack{1 < r < n \\ a+r+c=n}} (1^{\otimes a} \otimes b_r^+ \otimes 1^{\otimes c}) \phi_{a+1+c}^+ \\ &\quad + \sum_{a+1+c=n} (1^{\otimes a} \otimes b_1^- \otimes 1^{\otimes c}) \tilde{\phi}_n : \mathcal{Q}_{k-1}^\perp \rightarrow s\mathcal{D}. \end{aligned}$$

The restriction $\lambda_n|_{\mathcal{N}_k}$ takes values in $s\mathcal{C}$. Indeed, for the first sum in the expression for λ_n this follows by the induction assumption since $r > 1$ and $a + 1 + c > 1$. For the second sum this follows by the induction assumption and strict unitality if $n > 2$. In the case of $n = 2$, $k = 1$ this is also straightforward. The only case which requires computation is $n = 2$, $k = 2$:

$$(\mathbf{j}^c \otimes \mathbf{j}^c)(1 \otimes b_1^- + b_1^- \otimes 1) \tilde{b}_2 = \mathbf{j}^c - (\mathbf{j}^c \otimes \mathbf{i}_0^c) b_2^+ - \mathbf{j}^c - (\mathbf{i}_0^c \otimes \mathbf{j}^c) b_2^+,$$

which belongs to $s\mathcal{C}$ by the induction assumption.

Equations (3.2), (3.3) for $m = n$ take the form

$$-b_n^+ b_1 - \sum_{a+1+c=n} (1^{\otimes a} \otimes b_1' \otimes 1^{\otimes c}) b_n^+ = \lambda_n : \mathcal{N}_k \rightarrow s\mathcal{C}, \quad (3.4)$$

$$\phi_n^+ b_1 - \sum_{a+1+c=n} (1^{\otimes a} \otimes b_1' \otimes 1^{\otimes c}) \phi_n^+ - b_n^+ \phi_1 = \nu_n : \mathcal{N}_k \rightarrow s\mathcal{D}. \quad (3.5)$$

For arbitrary objects X, Y of \mathcal{C} , equip the graded \mathbb{k} -module $\mathcal{N}_k(X, Y)$ with the differential $d^{\mathcal{N}_k} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b_1' \otimes 1^{\otimes q}$ and denote by u the chain map

$$\begin{aligned} \underline{\mathcal{C}}_{\mathbb{k}}(\mathcal{N}_k(X, Y), s\mathcal{C}(X, Y)) &\rightarrow \underline{\mathcal{C}}_{\mathbb{k}}(\mathcal{N}_k(X, Y), s\mathcal{D}(X\phi, Y\phi)), \\ \lambda &\mapsto \lambda\phi_1. \end{aligned}$$

Since ϕ_1 is homotopy invertible, the map u is homotopy invertible as well. Therefore, the complex $\text{Cone}(u)$ is contractible, e.g. by [8, Lemma B.1], in particular, acyclic. Equations

(3.4) and (3.5) have the form $-b_n^+d = \lambda_n$, $\phi_n^+d + b_n^+u = \nu_n$, that is, the element (λ_n, ν_n) of

$$\underline{\mathbb{C}}_{\mathbb{k}}^2(\mathcal{N}_k(X, Y), s\mathcal{C}(X, Y)) \oplus \underline{\mathbb{C}}_{\mathbb{k}}^1(\mathcal{N}_k(X, Y), s\mathcal{D}(X\phi, Y\phi)) = \text{Cone}^1(u)$$

has to be the boundary of the sought element (b_n^+, ϕ_n^+) of

$$\underline{\mathbb{C}}_{\mathbb{k}}^1(\mathcal{N}_k(X, Y), s\mathcal{C}(X, Y)) \oplus \underline{\mathbb{C}}_{\mathbb{k}}^0(\mathcal{N}_k(X, Y), s\mathcal{D}(X\phi, Y\phi)) = \text{Cone}^0(u).$$

These equations are solvable because (λ_n, ν_n) is a cycle in $\text{Cone}^1(u)$. Indeed, the equations to verify $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ take the form

$$\begin{aligned} -\lambda_n b_1 + \sum_{p+1+q=n} (1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q}) \lambda_n &= 0 : \mathcal{N}_k \rightarrow s\mathcal{C}, \\ \nu_n b_1 + \sum_{p+1+q=n} (1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q}) \nu_n - \lambda_n \phi_1 &= 0 : \mathcal{N}_k \rightarrow s\mathcal{D}. \end{aligned}$$

Composing the identity $-\tilde{\lambda}b + \tilde{b}\lambda = 0 : T^n s\mathcal{C}^+ \rightarrow Ts\mathcal{C}^+$ with the projection $\text{pr}_1 : Ts\mathcal{C}^+ \rightarrow s\mathcal{C}^+$ yields the first equation. The second equation follows by composing the identity $\nu b^+ + \tilde{b}\nu - \lambda\tilde{\phi} = 0 : T^n s\mathcal{C}^+ \rightarrow Ts\mathcal{D}^+$ with $\text{pr}_1 : Ts\mathcal{D}^+ \rightarrow s\mathcal{D}^+$.

Thus, the required restrictions of b_n^+ , ϕ_n^+ to \mathcal{N}_k (and to \mathcal{Q}_k) exist and satisfy the required equations. We proceed by induction increasing k from 0 to n and determining b_n^+ , ϕ_n^+ on the whole $\mathcal{Q}_n = T^n s\mathcal{C}^+$. Then we replace n with $n+1$ and start again from $T^{n+1} s\mathcal{C}$. Thus the induction on n goes through. \square

3.8. Remark. Let (\mathcal{C}^+, b^+) be a homotopy unital structure of an A_∞ -category \mathcal{C} . Then the embedding A_∞ -functor $\iota : \mathcal{C} \rightarrow \mathcal{C}^+$ is an equivalence. Indeed, it is bijective on objects. By [8, Theorem 8.8] it suffices to prove that $\iota_1 : s\mathcal{C} \rightarrow s\mathcal{C}^+$ is homotopy invertible. And indeed, the chain quiver map $\pi_1 : s\mathcal{C}^+ \rightarrow s\mathcal{C}$, $\pi_1|_{s\mathcal{C}} = \text{id}$, ${}_X \mathbf{i}_0^{\text{csu}} \pi_1 = {}_X \mathbf{i}_0^{\mathcal{C}}$, $\mathbf{j}_X^{\mathcal{C}} \pi_1 = 0$, is homotopy inverse to ι_1 . Namely, the homotopy $h : s\mathcal{C}^+ \rightarrow s\mathcal{C}^+$, $h|_{s\mathcal{C}} = 0$, ${}_X \mathbf{i}_0^{\text{csu}} h = \mathbf{j}_X^{\mathcal{C}}$, $\mathbf{j}_X^{\mathcal{C}} h = 0$, satisfies the equation $\text{id}_{s\mathcal{C}^+} - \pi_1 \cdot \iota_1 = hb_1^+ + b_1^+ h$.

The equation between A_∞ -functors

$$[\mathcal{C} \xrightarrow{\iota^{\mathcal{C}}} \mathcal{C}^+ \xrightarrow{\phi^+} \mathcal{D}^+] = [\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\iota^{\mathcal{D}}} \mathcal{D}^+]$$

obtained in the proof of Theorem 3.7 implies that ϕ^+ is an A_∞ -equivalence as well. In particular, ϕ_1^+ is homotopy invertible.

The converse of Proposition 3.4 holds true as well, however its proof requires more preliminaries. It is deferred until Section 5.

4. DOUBLE CODERIVATIONS

4.1. Definition. For A_∞ -functors $f, g : \mathcal{A} \rightarrow \mathcal{B}$, a *double (f, g) -coderivation* of degree d is a system of \mathbb{k} -linear maps

$$r : (Ts\mathcal{A} \otimes Ts\mathcal{A})(X, Y) \rightarrow Ts\mathcal{B}(Xf, Yg), \quad X, Y \in \text{Ob } \mathcal{A},$$

of degree d such that the equation

$$r\Delta_0 = (\Delta_0 \otimes 1)(f \otimes r) + (1 \otimes \Delta_0)(r \otimes g) \tag{4.1}$$

holds true.

Equation (4.1) implies that r is determined by a system of \mathbb{k} -linear maps $r \operatorname{pr}_1 : Ts\mathcal{A} \otimes Ts\mathcal{A} \rightarrow s\mathcal{B}$ with components of degree d

$$r_{n,m} : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{n+m-1}, X_{n+m}) \rightarrow s\mathcal{B}(X_0 f, X_{n+m} g),$$

for $n, m \geq 0$, via the formula

$$\begin{aligned} r_{n,m;k} &= (r|_{T^n s\mathcal{A} \otimes T^m s\mathcal{A}}) \operatorname{pr}_k : T^n s\mathcal{A} \otimes T^m s\mathcal{A} \rightarrow T^k s\mathcal{B}, \\ r_{n,m;k} &= \sum_{\substack{p+1+q=k \\ i_1+\cdots+i_p+i=n, \\ j_1+\cdots+j_q+j=m}} f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \cdots \otimes g_{j_q}. \end{aligned} \quad (4.2)$$

This follows from the equation

$$r\Delta_0^{(l)} = \sum_{p+1+q=l} (\Delta_0^{(p+1)} \otimes \Delta_0^{(q+1)})(f^{\otimes p} \otimes r \otimes g^{\otimes q}) : Ts\mathcal{A} \otimes Ts\mathcal{A} \rightarrow (Ts\mathcal{B})^{\otimes l}, \quad (4.3)$$

which holds true for each $l \geq 0$. Here $\Delta_0^{(0)} = \varepsilon$, $\Delta_0^{(1)} = \operatorname{id}$, $\Delta_0^{(2)} = \Delta_0$ and $\Delta_0^{(l)}$ means the cut comultiplication iterated $l-1$ times.

Double (f, g) -coderivations form a chain complex, which we are going to denote by $(\mathcal{D}(\mathcal{A}, \mathcal{B})(f, g), B_1)$. For each $d \in \mathbb{Z}$, the component $\mathcal{D}(\mathcal{A}, \mathcal{B})(f, g)^d$ consists of double (f, g) -coderivations of degree d . The differential B_1 of degree 1 is given by

$$rB_1 \stackrel{\text{def}}{=} rb - (-)^d(1 \otimes b + b \otimes 1)r,$$

for each $r \in \mathcal{D}(\mathcal{A}, \mathcal{B})(f, g)^d$. The component $[rB_1]_{n,m}$ of rB_1 is given by

$$\begin{aligned} & \sum_{\substack{i_1+\cdots+i_p+i=n, \\ j_1+\cdots+j_q+j=m}} (f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \cdots \otimes g_{j_q}) b_{p+1+q} \\ & - (-)^r \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m}) r_{a+1+c,m} \\ & - (-)^r \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v}) r_{n,u+1+v}, \end{aligned} \quad (4.4)$$

for each $n, m \geq 0$. An A_∞ -functor $h : \mathcal{B} \rightarrow \mathcal{C}$ gives rise to a chain map

$$\mathcal{D}(\mathcal{A}, \mathcal{B})(f, g) \rightarrow \mathcal{D}(\mathcal{A}, \mathcal{C})(fh, gh), \quad r \mapsto rh.$$

The component $[rh]_{n,m}$ of rh is given by

$$\sum_{\substack{i_1+\cdots+i_p+i=n, \\ j_1+\cdots+j_q+j=m}} (f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \cdots \otimes g_{j_q}) h_{p+1+q}, \quad (4.5)$$

for each $n, m \geq 0$. Similarly, an A_∞ -functor $k : \mathcal{D} \rightarrow \mathcal{A}$ gives rise to a chain map

$$\mathcal{D}(\mathcal{A}, \mathcal{B})(f, g) \rightarrow \mathcal{D}(\mathcal{D}, \mathcal{B})(kf, kg), \quad r \mapsto (k \otimes k)r.$$

The component $[(k \otimes k)r]_{n,m}$ of $(k \otimes k)r$ is given by

$$\sum_{\substack{i_1+\dots+i_p=n \\ j_1+\dots+j_q=m}} (k_{i_1} \otimes \dots \otimes k_{i_p} \otimes k_{j_1} \otimes \dots \otimes k_{j_q}) r_{p,q}, \quad (4.6)$$

for each $n, m \geq 0$. Proofs of these facts are elementary and are left to the reader.

Let \mathcal{C} be an A_∞ -category. For each $n \geq 0$, introduce a morphism

$$\nu_n = \sum_{i=0}^n (-)^{n-i} (1^{\otimes i} \otimes \varepsilon \otimes 1^{\otimes n-i}) : (Ts\mathcal{C})^{\otimes n+1} \rightarrow (Ts\mathcal{C})^{\otimes n},$$

in $\mathcal{Q}/\text{Ob } \mathcal{C}$. In particular, $\nu_0 = \varepsilon : Ts\mathcal{C} \rightarrow \mathbb{k} \text{Ob } \mathcal{C}$. Denote $\nu = \nu_1 = (1 \otimes \varepsilon) - (\varepsilon \otimes 1) : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C}$ for the sake of brevity.

4.2. Lemma. *The map $\nu : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C}$ is a double $(1, 1)$ -coderivation of degree 0 and $\nu B_1 = 0$.*

Proof. We have:

$$\begin{aligned} & (\Delta_0 \otimes 1)(1 \otimes \nu) + (1 \otimes \Delta_0)(\nu \otimes 1) \\ &= (\Delta_0 \otimes 1)(1 \otimes 1 \otimes \varepsilon) - (\Delta_0 \otimes 1)(1 \otimes \varepsilon \otimes 1) \\ & \quad + (1 \otimes \Delta_0)(1 \otimes \varepsilon \otimes 1) - (1 \otimes \Delta_0)(\varepsilon \otimes 1 \otimes 1) \\ &= (\Delta_0 \otimes \varepsilon) - (\varepsilon \otimes \Delta_0) = ((1 \otimes \varepsilon) - (\varepsilon \otimes 1))\Delta_0 = \nu\Delta_0, \end{aligned}$$

due to the identities

$$(\Delta_0 \otimes 1)(1 \otimes \varepsilon \otimes 1) = 1 \otimes 1 = (1 \otimes \Delta_0)(1 \otimes \varepsilon \otimes 1) : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C} \otimes Ts\mathcal{C}.$$

This computation shows that $\nu : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C}$ is a double $(1, 1)$ -coderivation. Its only non-vanishing components are ${}_{X,Y}\nu_{1,0} = 1 : s\mathcal{C}(X, Y) \rightarrow s\mathcal{C}(X, Y)$ and ${}_{X,Y}\nu_{0,1} = 1 : s\mathcal{C}(X, Y) \rightarrow s\mathcal{C}(X, Y)$, $X, Y \in \text{Ob } \mathcal{C}$.

Since νB_1 is a double $(1, 1)$ -coderivation of degree 1, the equation $\nu B_1 = 0$ is equivalent to its particular case $\nu B_1 \text{pr}_1 = 0$, i.e., for each $n, m \geq 0$

$$\begin{aligned} & \sum_{\substack{0 \leq i \leq n, \\ 0 \leq j \leq m}} (1^{\otimes n-i} \otimes \nu_{i,j} \otimes 1^{\otimes m-j}) b_{n-i+1+m-j} - \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m}) \nu_{a+1+c,m} \\ & \quad - \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v}) \nu_{n,u+1+v} = 0 : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \rightarrow s\mathcal{C}. \end{aligned}$$

It reduces to the identity

$$\chi(n > 0) b_{n+m} - \chi(m > 0) b_{n+m} - \chi(m = 0) b_n + \chi(n = 0) b_m = 0,$$

where $\chi(P) = 1$ if a condition P holds and $\chi(P) = 0$ if P does not hold. \square

Let \mathcal{C} be a strictly unital A_∞ -category. The strict unit $\mathbf{i}_0^{\mathcal{C}}$ is viewed as a morphism of graded quivers $\mathbf{i}_0^{\mathcal{C}} : \mathbb{k} \text{Ob } \mathcal{C} \rightarrow s\mathcal{C}$ of degree -1 , identity on objects. For each $n \geq 0$, introduce a morphism of graded quivers

$$\xi_n = [(Ts\mathcal{C})^{\otimes n+1} \xrightarrow{1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1 \otimes \dots \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1} Ts\mathcal{C} \otimes s\mathcal{C} \otimes Ts\mathcal{C} \otimes \dots \otimes s\mathcal{C} \otimes Ts\mathcal{C} \xrightarrow{\mu^{(2n+1)}} Ts\mathcal{C}],$$

of degree $-n$, identity on objects. Here $\mu^{(2n+1)}$ denotes composition of $2n+1$ composable arrows in the graded category $Ts\mathcal{C}$. In particular, $\xi_0 = 1 : Ts\mathcal{C} \rightarrow Ts\mathcal{C}$. Denote $\xi = \xi_1 = (1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)} : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C}$ for the sake of brevity.

4.3. Lemma. *The map $\xi : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C}$ is a double $(1, 1)$ -coderivation of degree -1 and $\xi B_1 = \nu$.*

Proof. The following identity follows directly from the definitions of μ and Δ_0 :

$$\mu\Delta_0 = (\Delta_0 \otimes 1)(1 \otimes \mu) + (1 \otimes \Delta_0)(\mu \otimes 1) - 1 : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C} \otimes Ts\mathcal{C}.$$

It implies

$$\begin{aligned} \mu^{(3)}\Delta_0 &= (\Delta_0 \otimes 1 \otimes 1)(1 \otimes \mu^{(3)}) + (1 \otimes 1 \otimes \Delta_0)(\mu^{(3)} \otimes 1) \\ &\quad + (1 \otimes \Delta_0 \otimes 1)(\mu \otimes \mu) - (1 \otimes \mu) - (\mu \otimes 1) : Ts\mathcal{C} \otimes Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C} \otimes Ts\mathcal{C}. \end{aligned} \quad (4.7)$$

Since $\mathbf{i}_0^{\mathcal{C}}\Delta_0 = \mathbf{i}_0^{\mathcal{C}} \otimes \eta + \eta \otimes \mathbf{i}_0^{\mathcal{C}} : \mathbb{k} \text{Ob } \mathcal{C} \rightarrow Ts\mathcal{C} \otimes Ts\mathcal{C}$, it follows that

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}}\Delta_0 \otimes 1)(\mu \otimes \mu) - (1 \otimes (\mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu) - ((1 \otimes \mathbf{i}_0^{\mathcal{C}})\mu \otimes 1) = 0 : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C} \otimes Ts\mathcal{C}.$$

Equation (4.7) yields

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)}\Delta_0 = (\Delta_0 \otimes 1)(1 \otimes (1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)}) + (1 \otimes \Delta_0)((1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)} \otimes 1),$$

i.e., $\xi = (1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)} : Ts\mathcal{C} \otimes Ts\mathcal{C} \rightarrow Ts\mathcal{C}$ is a double $(1, 1)$ -coderivation. Its the only non-vanishing components are ${}_X\xi_{0,0} = {}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X, X)$, $X \in \text{Ob } \mathcal{C}$.

Since both ξB_1 and ν are double $(1, 1)$ -coderivations of degree 0, the equation $\xi B_1 = \nu$ is equivalent to its particular case $\xi B_1 \text{pr}_1 = \nu \text{pr}_1$, i.e., for each $n, m \geq 0$

$$\begin{aligned} \sum_{\substack{0 \leq p \leq n \\ 0 \leq q \leq m}} (1^{\otimes n-p} \otimes \xi_{p,q} \otimes 1^{\otimes m-q})b_{n-p+1+m-q} + \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m})\xi_{a+1+c,m} \\ + \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v})\xi_{n,u+1+v} = \nu_{n,m} : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \rightarrow s\mathcal{C}. \end{aligned}$$

It reduces to the the equation

$$(1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes m})b_{n+1+m} = \nu_{n,m} : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \rightarrow s\mathcal{C},$$

which holds true, since $\mathbf{i}_0^{\mathcal{C}}$ is a strict unit. □

Note that the maps ν_n, ξ_n obey the following relations:

$$\xi_n = (\xi_{n-1} \otimes 1)\xi, \quad \nu_n = (1^{\otimes n} \otimes \varepsilon) - (\nu_{n-1} \otimes 1), \quad n \geq 1. \quad (4.8)$$

In particular, $\xi_n \varepsilon = 0 : (Ts\mathcal{C})^{\otimes n+1} \rightarrow \mathbb{k} \text{Ob } \mathcal{C}$, for each $n \geq 1$, as $\xi \varepsilon = 0$ by equation (4.3).

4.4. **Lemma.** *The following equations hold true:*

$$\xi_n \Delta_0 = \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (\xi_i \otimes \xi_{n-i}), \quad n \geq 0, \quad (4.9)$$

$$\xi_n b - (-)^n \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \xi_n = \nu_n \xi_{n-1}, \quad n \geq 1. \quad (4.10)$$

Proof. Let us prove (4.9). The proof is by induction on n . The case $n = 0$ is trivial. Let $n \geq 1$. By (4.8) and Lemma 4.3,

$$\xi_n \Delta_0 = (\xi_{n-1} \otimes 1) \xi \Delta_0 = (\xi_{n-1} \Delta_0 \otimes 1) (1 \otimes \xi) + (\xi_{n-1} \otimes \Delta_0) (\xi \otimes 1).$$

By induction hypothesis,

$$\xi_{n-1} \Delta_0 = \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-1-i}) (\xi_i \otimes \xi_{n-1-i}),$$

therefore

$$\begin{aligned} \xi_n \Delta_0 &= \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (\xi_i \otimes \xi_{n-1-i} \otimes 1) (1 \otimes \xi) + (1^{\otimes n} \otimes \Delta_0) ((\xi_{n-1} \otimes 1) \xi \otimes 1) \\ &= \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (\xi_i \otimes \xi_{n-i}), \end{aligned}$$

since $(\xi_{n-1-i} \otimes 1) \xi = \xi_{n-i}$ if $0 \leq i \leq n-1$.

Let us prove (4.10). The proof is by induction on n . The case $n = 1$ follows from Lemma 4.3. Let $n \geq 2$. By (4.8) and Lemma 4.3,

$$\begin{aligned} \xi_n b - (-)^n \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \xi_n &= (\xi_{n-1} \otimes 1) \xi b - (-)^n \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \xi_{n-1} \otimes 1) \xi - (-)^n (1^{\otimes n} \otimes b) (\xi_{n-1} \otimes 1) \xi \\ &= -(\xi_{n-1} b \otimes 1) \xi - (\xi_{n-1} \otimes b) \xi + (\xi_{n-1} \otimes 1) \nu \\ &\quad + (-)^{n-1} \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \xi_{n-1} \otimes 1) \xi + (\xi_{n-1} \otimes b) \xi \\ &= (\xi_{n-1} \otimes 1) \nu - \left(\left[\xi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \xi_{n-1} \right] \otimes 1 \right) \xi. \end{aligned}$$

By induction hypothesis

$$\xi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \xi_{n-1} = \nu_{n-1} \xi_{n-2},$$

therefore

$$\xi_n b - (-)^n \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \xi_n = (\xi_{n-1} \otimes 1) \nu - (\nu_{n-1} \xi_{n-2} \otimes 1) \xi.$$

Since by (4.8),

$$\begin{aligned} (\xi_{n-1} \otimes 1) \nu - (\nu_{n-1} \xi_{n-2} \otimes 1) \xi &= (\xi_{n-1} \otimes \varepsilon) - (\xi_{n-1} \varepsilon \otimes 1) - (\nu_{n-1} \otimes 1) \xi_{n-1} \\ &= (1^{\otimes n} \otimes \varepsilon) \xi_{n-1} - (\nu_{n-1} \otimes 1) \xi_{n-1} = \nu_n \xi_{n-1}, \end{aligned}$$

equation (4.10) is proven. \square

5. AN AUGMENTED DIFFERENTIAL GRADED COCATEGORY

Let now $\mathcal{C} = \mathcal{A}^{\text{su}}$, where \mathcal{A} is an A_∞ -category. There is an isomorphism of graded \mathbb{k} -quivers, identity on objects:

$$\zeta : \bigoplus_{n \geq 0} (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{A}^{\text{su}}.$$

The morphism ζ is the sum of morphisms

$$\zeta_n = [(Ts\mathcal{A})^{\otimes n+1}[n] \xrightarrow{s^{-n}} (Ts\mathcal{A})^{\otimes n+1} \xrightarrow{e^{\otimes n+1}} (Ts\mathcal{A}^{\text{su}})^{\otimes n+1} \xrightarrow{\xi_n} Ts\mathcal{A}^{\text{su}}], \quad (5.1)$$

where $e : \mathcal{A} \hookrightarrow \mathcal{A}^{\text{su}}$ is the natural embedding. The graded quiver

$$\mathcal{E} \stackrel{\text{def}}{=} \bigoplus_{n \geq 0} (Ts\mathcal{A})^{\otimes n+1}[n]$$

admits a unique structure of an augmented differential graded cocategory such that ζ becomes an isomorphism of augmented differential graded cocategories. The comultiplication $\tilde{\Delta} : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{E}$ is found from the equation

$$[\mathcal{E} \xrightarrow{\zeta} Ts\mathcal{A}^{\text{su}} \xrightarrow{\Delta_0} Ts\mathcal{A}^{\text{su}} \otimes Ts\mathcal{A}^{\text{su}}] = [\mathcal{E} \xrightarrow{\tilde{\Delta}} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\zeta \otimes \zeta} Ts\mathcal{A}^{\text{su}} \otimes Ts\mathcal{A}^{\text{su}}].$$

Restricting the left hand side of the equation to the summand $(Ts\mathcal{A})^{\otimes n+1}[n]$ of \mathcal{E} , we obtain

$$\begin{aligned} \zeta_n \Delta_0 &= s^{-n} e^{\otimes n+1} \xi_n \Delta_0 \\ &= s^{-n} \sum_{i=0}^n (e^{\otimes i} \otimes e \Delta_0 \otimes e^{\otimes n-i}) (\xi_i \otimes \xi_{n-i}) : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{A}^{\text{su}} \otimes Ts\mathcal{A}^{\text{su}}, \end{aligned}$$

by equation (4.9). Since e is a morphism of augmented graded cocategories, it follows that

$$\begin{aligned} \zeta_n \Delta_0 &= s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (e^{\otimes i+1} \xi_i \otimes e^{\otimes n-i+1} \xi_{n-i}) \\ &= s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (s^i \otimes s^{n-i}) (\zeta_i \otimes \zeta_{n-i}) : \\ &\quad (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{A}^{\text{su}} \otimes Ts\mathcal{A}^{\text{su}}. \end{aligned}$$

This implies the following formula for $\tilde{\Delta}$:

$$\begin{aligned} \tilde{\Delta}|_{(Ts\mathcal{A})^{\otimes n+1}[n]} &= s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(s^i \otimes s^{n-i}) : \\ &(Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow \bigoplus_{i=0}^n (Ts\mathcal{A})^{\otimes i+1}[i] \bigotimes (Ts\mathcal{A})^{\otimes n-i+1}[n-i]. \end{aligned} \quad (5.2)$$

The counit of \mathcal{E} is $\tilde{\varepsilon} = [\mathcal{E} \xrightarrow{\text{pr}_0} Ts\mathcal{A} \xrightarrow{\varepsilon} \mathbb{k} \text{Ob } \mathcal{A} = \mathbb{k} \text{Ob } \mathcal{E}]$. The augmentation of \mathcal{E} is $\tilde{\eta} = [\mathbb{k} \text{Ob } \mathcal{E} = \mathbb{k} \text{Ob } \mathcal{A} \xrightarrow{\eta} Ts\mathcal{A} \xrightarrow{\text{in}_0} \mathcal{E}]$. The differential $\tilde{b} : \mathcal{E} \rightarrow \mathcal{E}$ is found from the following equation:

$$[\mathcal{E} \xrightarrow{\zeta} Ts\mathcal{A}^{\text{su}} \xrightarrow{b} Ts\mathcal{A}^{\text{su}}] = [\mathcal{E} \xrightarrow{\tilde{b}} \mathcal{E} \xrightarrow{\zeta} Ts\mathcal{A}^{\text{su}}].$$

Let $\tilde{b}_{n,m} : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow (Ts\mathcal{A})^{\otimes m+1}[m]$, $n, m \geq 0$, denote the matrix coefficients of \tilde{b} . Restricting the left hand side of the above equation to the summand $(Ts\mathcal{A})^{\otimes n+1}[n]$ of \mathcal{E} , we obtain

$$\begin{aligned} \zeta_n b &= s^{-n} e^{\otimes n+1} \xi_n b \\ &= s^{-n} e^{\otimes n+1} \nu_n \xi_{n-1} + (-)^n s^{-n} \sum_{i=0}^n (e^{\otimes i} \otimes eb \otimes e^{\otimes n-i}) \xi_n : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{A}^{\text{su}}, \end{aligned}$$

by equation (4.10). Since e preserves the counit, it follows that

$$e^{\otimes n+1} \nu_n = \nu_n e^{\otimes n} : (Ts\mathcal{A})^{\otimes n+1} \rightarrow (Ts\mathcal{A}^{\text{su}})^{\otimes n}.$$

Furthermore, e commutes with the differential b , therefore

$$\begin{aligned} \zeta_n b &= s^{-n} \nu_n s^{n-1} (s^{-(n-1)} e^{\otimes n} \xi_{n-1}) + (-)^n s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) s^n (s^{-n} e^{\otimes n+1} \xi_n) \\ &= s^{-n} \nu_n s^{n-1} \zeta_{n-1} + (-)^n s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) s^n \zeta_n : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{A}^{\text{su}}. \end{aligned}$$

We conclude that

$$\tilde{b}_{n,n} = (-)^n s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) s^n : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow (Ts\mathcal{A})^{\otimes n+1}[n], \quad (5.3)$$

for $n \geq 0$, and

$$\tilde{b}_{n,n-1} = s^{-n} \nu_n s^{n-1} : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow (Ts\mathcal{A})^{\otimes n}[n-1], \quad (5.4)$$

for $n \geq 1$, are the only non-vanishing matrix coefficients of \tilde{b} .

Let $g : \mathcal{E} \rightarrow Ts\mathcal{B}$ be a morphism of augmented differential graded cocategories, and let $g_n : (Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{B}$ be its components. By formula (5.2), the equation $g\Delta_0 =$

$\tilde{\Delta}(g \otimes g)$ is equivalent to the system of equations

$$g_n \Delta_0 = s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (s^i g_i \otimes s^{n-i} g_{n-i}) :$$

$$(Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{B} \otimes Ts\mathcal{B}, \quad n \geq 0.$$

The equation $g\varepsilon = \tilde{\varepsilon}(\mathbb{k} \text{Ob } g)$ is equivalent to the equations $g_0\varepsilon = \varepsilon(\mathbb{k} \text{Ob } g_0)$, $g_n\varepsilon = 0$, $n \geq 1$. The equation $\tilde{\eta}g = (\mathbb{k} \text{Ob } g)\eta$ is equivalent to the equation $\eta g_0 = (\mathbb{k} \text{Ob } g_0)\eta$. By formulas (5.3) and (5.4), the equation $gb = \tilde{b}g$ is equivalent to $g_0b = bg_0 : Ts\mathcal{A} \rightarrow Ts\mathcal{B}$ and

$$g_nb = (-)^n s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) s^n g_n + s^{-n} \nu_n s^{n-1} g_{n-1} :$$

$$(Ts\mathcal{A})^{\otimes n+1}[n] \rightarrow Ts\mathcal{B}, \quad n \geq 1.$$

Introduce \mathbb{k} -linear maps $\phi_n = s^n g_n : (Ts\mathcal{A})^{\otimes n+1}(X, Y) \rightarrow Ts\mathcal{B}(Xg, Yg)$ of degree $-n$, $X, Y \in \text{Ob } \mathcal{A}$, $n \geq 0$. The above equations take the following form:

$$\phi_n \Delta_0 = \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (\phi_i \otimes \phi_{n-i}) : (Ts\mathcal{A})^{\otimes n+1} \rightarrow Ts\mathcal{B} \otimes Ts\mathcal{B}, \quad (5.5)$$

for $n \geq 1$;

$$\phi_nb = (-)^n \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n + \nu_n \phi_{n-1} : (Ts\mathcal{A})^{\otimes n+1} \rightarrow Ts\mathcal{B}, \quad (5.6)$$

for $n \geq 1$;

$$\phi_0 \Delta_0 = \Delta_0(\phi_0 \otimes \phi_0), \quad \phi_0 \varepsilon = \varepsilon, \quad \phi_0 b = b\phi_0, \quad (5.7)$$

$$\phi_n \varepsilon = 0, \quad n \geq 1. \quad (5.8)$$

Summing up, we conclude that morphisms of augmented differential graded cocategories $\mathcal{E} \rightarrow Ts\mathcal{B}$ are in bijection with collections consisting of a morphism of augmented differential graded cocategories $\phi_0 : Ts\mathcal{A} \rightarrow Ts\mathcal{B}$ and of \mathbb{k} -linear maps $\phi_n : (Ts\mathcal{A})^{\otimes n+1}(X, Y) \rightarrow Ts\mathcal{B}(X\phi_0, Y\phi_0)$ of degree $-n$, $X, Y \in \text{Ob } \mathcal{A}$, $n \geq 1$, such that equations (5.5), (5.6), and (5.8) hold true.

In particular, A_∞ -functors $f : \mathcal{A}^{\text{su}} \rightarrow \mathcal{B}$, which are augmented differential graded cocategory morphisms $Ts\mathcal{A}^{\text{su}} \rightarrow Ts\mathcal{B}$, are in bijection with morphisms $g = \zeta f : \mathcal{E} \rightarrow Ts\mathcal{B}$ of augmented differential graded cocategories. With the above notation, we may say that to give an A_∞ -functor $f : \mathcal{A}^{\text{su}} \rightarrow \mathcal{B}$ is the same as to give an A_∞ -functor $\phi_0 : \mathcal{A} \rightarrow \mathcal{B}$ and a system of \mathbb{k} -linear maps $\phi_n : (Ts\mathcal{A})^{\otimes n+1}(X, Y) \rightarrow Ts\mathcal{B}(X\phi_0, Y\phi_0)$ of degree $-n$, $X, Y \in \text{Ob } \mathcal{A}$, $n \geq 1$, such that equations (5.5), (5.6) and (5.8) hold true.

5.1. Proposition. *The following conditions are equivalent.*

(a) *There exists an A_∞ -functor $U : \mathcal{A}^{\text{su}} \rightarrow \mathcal{A}$ such that*

$$[\mathcal{A} \xrightarrow{e} \mathcal{A}^{\text{su}} \xrightarrow{U} \mathcal{A}] = \text{id}_{\mathcal{A}}.$$

- (b) *There exists a double $(1, 1)$ -coderivation $\phi : Ts\mathcal{A} \otimes Ts\mathcal{A} \rightarrow Ts\mathcal{A}$ of degree -1 such that $\phi B_1 = \nu$.*

Proof. (a) \Rightarrow (b) Let $U : \mathcal{A}^{\text{su}} \rightarrow \mathcal{A}$ be an A_∞ -functor such that $eU = \text{id}_{\mathcal{A}}$, in particular $\text{Ob } U = \text{id} : \text{Ob } \mathcal{A}^{\text{su}} = \text{Ob } \mathcal{A} \rightarrow \text{Ob } \mathcal{A}$. It gives rise to the family of \mathbb{k} -linear maps $\phi_n = s^n \zeta_n U : (Ts\mathcal{A})^{\otimes n+1}(X, Y) \rightarrow Ts\mathcal{B}(X, Y)$ of degree $-n$, $X, Y \in \text{Ob } \mathcal{A}$, $n \geq 0$, that satisfy equations (5.5), (5.6) and (5.8). In particular, $\phi_0 = eU = \text{id}_{\mathcal{A}}$. Equations (5.5) and (5.6) for $n = 1$ read as follows:

$$\begin{aligned}\phi_1 \Delta_0 &= (\Delta_0 \otimes 1)(\phi_0 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes \phi_0) = (\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes 1), \\ \phi_1 b &= (1 \otimes b + b \otimes 1)\phi_1 + \nu_1 \phi_0 = (1 \otimes b + b \otimes 1)\phi_1 + \nu.\end{aligned}$$

In other words, ϕ_1 is a double $(1, 1)$ -coderivation of degree -1 and $\phi_1 B_1 = \nu$.

(b) \Rightarrow (a) Let $\phi : Ts\mathcal{A} \otimes Ts\mathcal{A} \rightarrow Ts\mathcal{A}$ be a double $(1, 1)$ -coderivation of degree -1 such that $\phi B_1 = \nu$. Define \mathbb{k} -linear maps

$$\phi_n : (Ts\mathcal{A})^{\otimes n+1}(X, Y) \rightarrow Ts\mathcal{A}(X, Y), \quad X, Y \in \text{Ob } \mathcal{A},$$

of degree $-n$, $n \geq 0$, recursively via $\phi_0 = \text{id}_{\mathcal{A}}$ and $\phi_n = (\phi_{n-1} \otimes 1)\phi$, $n \geq 1$. Let us show that ϕ_n satisfy equations (5.5), (5.6) and (5.8). Equation (5.8) is obvious: $\phi_n \varepsilon = (\phi_{n-1} \otimes 1)\phi \varepsilon = 0$ as $\phi \varepsilon = 0$ by (4.3). Let us prove equation (5.5) by induction. It holds for $n = 1$ by assumption, since $\phi_1 = \phi$ is a double $(1, 1)$ -coderivation. Let $n \geq 2$. We have:

$$\begin{aligned}\phi_n \Delta_0 &= (\phi_{n-1} \otimes 1)\phi_1 \Delta_0 \\ &= (\phi_{n-1} \otimes 1)((\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes 1)) \\ &= (\phi_{n-1} \Delta_0 \otimes 1)(1 \otimes \phi_1) + (1^{\otimes n} \otimes \Delta_0)((\phi_{n-1} \otimes 1)\phi_1 \otimes 1).\end{aligned}$$

By induction hypothesis,

$$\phi_{n-1} \Delta_0 = \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-1-i})(\phi_i \otimes \phi_{n-1-i}),$$

so that

$$\begin{aligned}\phi_n \Delta_0 &= \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\phi_i \otimes \phi_{n-1-i} \otimes 1)(1 \otimes \phi_1) \\ &\quad + (1^{\otimes n} \otimes \Delta_0)((\phi_{n-1} \otimes 1)\phi_1 \otimes 1) \\ &= \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\phi_i \otimes \phi_{n-i}),\end{aligned}$$

since $(\phi_{n-1-i} \otimes 1)\phi_1 = \phi_{n-i}$, $0 \leq i \leq n-1$.

Let us prove equation (5.6) by induction. For $n = 1$ it is equivalent to the equation $\phi B_1 = \nu$, which holds by assumption. Let $n \geq 2$. We have:

$$\begin{aligned}
& \phi_n b - (-)^n \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n \\
&= (\phi_{n-1} \otimes 1) \phi b - (-)^n \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} \otimes 1) \phi \\
&\quad - (-)^n (1^{\otimes n} \otimes b) (\phi_{n-1} \otimes 1) \phi \\
&= -(\phi_{n-1} b \otimes 1) \phi - (\phi_{n-1} \otimes b) \phi + (\phi_{n-1} \otimes 1) \nu \\
&\quad + (-)^{n-1} \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} \otimes 1) \phi + (\phi_{n-1} \otimes b) \phi \\
&= (\phi_{n-1} \otimes 1) \nu - \left(\left[\phi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} \right] \otimes 1 \right) \phi.
\end{aligned}$$

By induction hypothesis,

$$\phi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} = \nu_{n-1} \phi_{n-2},$$

therefore

$$\phi_n b - (-)^n \sum_{i=0}^n (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n = (\phi_{n-1} \otimes 1) \nu - (\nu_{n-1} \phi_{n-2} \otimes 1) \phi.$$

Since by (4.8)

$$\begin{aligned}
(\phi_{n-1} \otimes 1) \nu - (\nu_{n-1} \phi_{n-2} \otimes 1) \phi &= (\phi_{n-1} \otimes \varepsilon) - (\phi_{n-1} \varepsilon \otimes 1) - (\nu_{n-1} \otimes 1) \phi_{n-1} \\
&= (1^{\otimes n} \otimes \varepsilon) \phi_{n-1} - (\nu_{n-1} \otimes 1) \phi_{n-1} = \nu_n \phi_{n-1},
\end{aligned}$$

and equation (5.6) is proven.

The system of maps ϕ_n , $n \geq 0$, corresponds to an A_∞ -functor $U : \mathcal{A}^{\text{su}} \rightarrow \mathcal{A}$ such that $\phi_n = s^n \zeta_n U$, $n \geq 0$. In particular, $eU = \phi_0 = \text{id}_{\mathcal{A}}$. \square

5.2. Proposition. *Let \mathcal{A} be a unital A_∞ -category. There exists a double $(1, 1)$ -coderivation $h : Ts\mathcal{A} \otimes Ts\mathcal{A} \rightarrow Ts\mathcal{A}$ of degree -1 such that $hB_1 = \nu$.*

Proof. Let \mathcal{A} be a unital A_∞ -category. By [10, Corollary A.12], there exist a differential graded category \mathcal{D} and an A_∞ -equivalence $f : \mathcal{A} \rightarrow \mathcal{D}$. The functor f is unital by [8, Corollary 8.9]. This means that, for every object X of \mathcal{A} , there exists a \mathbb{k} -linear map ${}_X v_0 : \mathbb{k} \rightarrow (s\mathcal{D})^{-2}(Xf, Xf)$ such that ${}_X \mathbf{i}_0^{\mathcal{A}} f_1 = {}_X f \mathbf{i}_0^{\mathcal{D}} + {}_X v_0 b_1$. Here ${}_X f \mathbf{i}_0^{\mathcal{D}}$ denotes the strict unit of the differential graded category \mathcal{D} .

By Lemma 4.3, $\xi = (1 \otimes \mathbf{i}_0^{\mathcal{D}} \otimes 1) \mu^{(3)} : Ts\mathcal{D} \otimes Ts\mathcal{D} \rightarrow Ts\mathcal{D}$ is a $(1, 1)$ -coderivation of degree -1 . Let ι denote the double (f, f) -coderivation $(f \otimes f) \xi$ of degree -1 . By Lemma 4.3,

$$\iota B_1 = (f \otimes f)(\xi B_1) = (f \otimes f) \nu = \nu f.$$

By Lemma 4.2, the equation $\nu B_1 = 0$ holds true. We conclude that the double coderivations $\nu \in \mathcal{D}(\mathcal{A}, \mathcal{A})(\text{id}_{\mathcal{A}}, \text{id}_{\mathcal{A}})^0$ and $\iota \in \mathcal{D}(\mathcal{A}, \mathcal{D})(f, f)^{-1}$ satisfy the following equations:

$$\nu B_1 = 0, \quad (5.9)$$

$$\iota B_1 - \nu f = 0. \quad (5.10)$$

We are going to prove that there exist double coderivations $h \in \mathcal{D}(\mathcal{A}, \mathcal{A})(\text{id}_{\mathcal{A}}, \text{id}_{\mathcal{A}})^{-1}$ and $k \in \mathcal{D}(\mathcal{A}, \mathcal{D})(f, f)^{-2}$ such that the following equations hold true:

$$\begin{aligned} h B_1 &= \nu, \\ h f &= \iota + k B_1. \end{aligned}$$

Let us put ${}_X h_{0,0} = {}_X \mathbf{i}_0^{\mathcal{A}}$, ${}_X k_{0,0} = {}_X v_0$, and construct the other components of h and k by induction. Given an integer $t \geq 0$, assume that we have already found components $h_{p,q}$, $k_{p,q}$ of the sought h , k , for all pairs (p, q) with $p + q < t$, such that the equations

$$(h B_1 - \nu)_{p,q} = 0 : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{p+q-1}, X_{p+q}) \rightarrow s\mathcal{A}(X_0, X_{p+q}), \quad (5.11)$$

$$(k B_1 + \iota - h f)_{p,q} = 0 : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{p+q-1}, X_{p+q}) \rightarrow s\mathcal{D}(X_0 f, X_{p+q} f) \quad (5.12)$$

are satisfied for all pairs (p, q) with $p + q < t$. Introduce double coderivations $\tilde{h} \in \mathcal{D}(\mathcal{A}, \mathcal{A})(\text{id}_{\mathcal{A}}, \text{id}_{\mathcal{A}})$ and $\tilde{k} \in \mathcal{D}(\mathcal{A}, \mathcal{D})(f, f)$ of degree -1 resp. -2 by their components: $\tilde{h}_{p,q} = h_{p,q}$, $\tilde{k}_{p,q} = k_{p,q}$ for $p+q < t$, all the other components vanish. Define a double $(1, 1)$ -coderivation $\lambda = \tilde{h} B_1 - \nu$ of degree 0 and a double (f, f) -coderivation $\kappa = \tilde{k} B_1 + \iota - \tilde{h} f$ of degree -1 . Then $\lambda_{p,q} = 0$, $\kappa_{p,q} = 0$ for all $p + q < t$. Let non-negative integers n, m satisfy $n + m = t$. The identity $\lambda B_1 = 0$ implies that

$$\lambda_{n,m} b_1 - \sum_{l=1}^{n+m} (1^{\otimes l-1} \otimes b_1 \otimes 1^{\otimes n+m-l}) \lambda_{n,m} = 0.$$

The (n, m) -component of the identity $\kappa B_1 + \lambda f = 0$ gives

$$\kappa_{n,m} b_1 + \sum_{l=1}^{n+m} (1^{\otimes l-1} \otimes b_1 \otimes 1^{\otimes n+m-l}) \kappa_{n,m} + \lambda_{n,m} f_1 = 0.$$

The chain map $f_1 : \mathcal{A}(X_0, X_{n+m}) \rightarrow s\mathcal{D}(X_0 f, X_{n+m} f)$ is homotopy invertible as f is an A_∞ -equivalence. Hence, the chain map Φ given by

$$\begin{aligned} \underline{\mathbb{C}}_{\mathbb{k}}^\bullet(N, s\mathcal{A}(X_0, X_{n+m})) &\rightarrow \underline{\mathbb{C}}_{\mathbb{k}}^\bullet(N, s\mathcal{D}(X_0 f, X_{n+m} f)), \\ \lambda &\mapsto \lambda f_1, \end{aligned}$$

is homotopy invertible for each complex of \mathbb{k} -modules N , in particular, for $N = s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{n+m-1}, X_{n+m})$. Therefore, the complex $\text{Cone}(\Phi)$ is contractible, e.g. by [8, Lemma B.1]. Consider the element $(\lambda_{n,m}, \kappa_{n,m})$ of

$$\underline{\mathbb{C}}_{\mathbb{k}}^0(N, s\mathcal{A}(X_0, X_{n+m})) \oplus \underline{\mathbb{C}}_{\mathbb{k}}^{-1}(N, \mathcal{D}(X_0 f, X_{n+m} f)).$$

The above direct sum coincides with $\text{Cone}^{-1}(\Phi)$. The equations $-\lambda_{n,m} d = 0$, $\kappa_{n,m} d + \lambda_{n,m} \Phi = 0$ imply that $(\lambda_{n,m}, \kappa_{n,m})$ is a cycle in the complex $\text{Cone}(\Phi)$. Due to acyclicity of

$\text{Cone}(\Phi)$, $(\lambda_{n,m}, \kappa_{n,m})$ is a boundary of some element $(h_{n,m}, -k_{n,m})$ of $\text{Cone}^{-2}(\Phi)$, i.e., of

$$\underline{\mathbb{C}}_{\mathbb{k}}^{-1}(N, s\mathcal{A}(X_0, X_{n+m})) \oplus \underline{\mathbb{C}}_{\mathbb{k}}^{-2}(N, \mathcal{D}(X_0 f, X_{n+m} f)).$$

Thus, $-k_{n,m}d + h_{n,m}f_1 = \kappa_{n,m}$, $-h_{n,m}d = \lambda_{n,m}$. These equations can be written as follows:

$$\begin{aligned} -h_{n,m}b_1 - \sum_{u+1+v=n+m} (1^{\otimes u} \otimes b_1 \otimes 1^{\otimes v})h_{n,m} &= (\tilde{h}B_1 - \nu)_{n,m}, \\ -k_{n,m}b_1 + \sum_{u+1+v=n+m} (1^{\otimes u} \otimes b_1 \otimes 1^{\otimes v})k_{n,m} + h_{n,m}f_1 &= (\tilde{k}B_1 + \iota - \tilde{h}f)_{n,m}. \end{aligned}$$

Thus, if we introduce double coderivations \bar{h} and \bar{k} by their components: $\bar{h}_{p,q} = h_{p,q}$, $\bar{k}_{p,q} = k_{p,q}$ for $p+q \leq t$ (using just found maps if $p+q = t$) and 0 otherwise, then these coderivations satisfy equations (5.11) and (5.12) for each p, q such that $p+q \leq t$. Induction on t proves the proposition. \square

5.3. Theorem. *Every unital A_∞ -category admits a weak unit.*

Proof. The proof follows from Propositions 5.1 and 5.2. \square

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